Operations With Rational Expressions

A *rational expression* is an expression that can be written in the form $\frac{\text{polynominal}}{\text{polynominal}}$, where the denominator is not zero. A rational expression is in simplest form if the numerator and denominator have no common factors except 1.

Example 1

Write the expression $\frac{4x + 8}{x + 2}$ in simplest form.

Factor the numerator. $\frac{4x + 8}{x + 2} = \frac{4(x + 2)}{x + 2}$

Divide out the common factor $x + 2$. $= 4$

To add or subtract two rational expressions, use a common denominator.

Example 2

Simplify $\frac{x}{2y} + \frac{x}{3y}$.

The common denominator of $3y$ and $2y$ is $6y$. $\frac{x}{2y} + \frac{x}{3y} = \frac{x}{2y} \cdot \frac{3}{3} + \frac{x}{3y} \cdot \frac{2}{2}$

Multiply. $= \frac{3x}{6y} + \frac{2x}{6y}$

Add the numerators. $= \frac{5x}{6y}$

To multiply rational expressions, first find and divide out any common factors in the numerators and the denominators. Then multiply the remaining numerators and denominators. To divide rational expressions, first use a reciprocal to change the problem to multiplication.

Example 3

Simplify $\frac{40x^2}{21} \div \frac{5x}{14}$.

Change dividing by $\frac{5x}{14}$ to multiplying by the reciprocal, $\frac{14}{5x}$. $\frac{40x^2}{21} \div \frac{5x}{14} = \frac{40x^2}{21} \cdot \frac{14}{5x}$

Divide out the common factors 5, x, and 7. $= \frac{{}^{8}\cancel{40}\cancel{x^2}{}^{1}}{{}_{3}\cancel{21}} \times \frac{\cancel{14}^{2}}{\cancel{5x}_{1}}$

Multiply the numerators ($8x \cdot 2$). Multiply the denominators ($3 \cdot 1$). $= \frac{16x}{3}$

Need help on rational expressions? Click here.

Exercises

Write each expression in simplest form.

1. $\frac{4a^2b}{12ab^3}$
2. $\frac{5n + 15}{n + 3}$
3. $\frac{x - 7}{2x - 14}$
4. $\frac{28c^2(d - 3)}{35c(d - 3)}$

Perform the indicated operation.

5. $\frac{3x}{2} + \frac{5x}{2}$
6. $\frac{3x}{8} + \frac{5x}{8}$
7. $\frac{5}{h} - \frac{3}{h}$
8. $\frac{6}{11p} - \frac{9}{11p}$
9. $\frac{3x}{5} - \frac{x}{2}$
10. $\frac{13}{2x} - \frac{13}{3x}$
11. $\frac{7x}{5} + \frac{5x}{7}$
12. $\frac{5a}{b} + \frac{3a}{5b}$
13. $\frac{7x}{8} \cdot \frac{32x}{35}$
14. $\frac{3x^2}{2} \cdot \frac{6}{x}$
15. $\frac{8x^2}{5} \cdot \frac{10}{x^3}$
16. $\frac{7x}{8} \cdot \frac{64}{14x}$
17. $\frac{16}{3x} \div \frac{5}{3x}$
18. $\frac{4x}{5} \div \frac{16}{15x}$
19. $\frac{x^3}{8} \div \frac{x^2}{16}$

Factoring and Operations With Polynomials

Example 1

Perform each operation.

a. $(3y^2 - 4y + 5) + (y^2 + 9y)$

To add, group like terms. $= (3y^2 + y^2) + (-4y + 9y) + 5$

Combine like terms. $= 4y^2 + 5y + 5$

b. $(n + 4)(n - 3)$

Distribute n and 4. $= n(n) + n(-3) + 4(n) + 4(-3)$

Multiply. $= n^2 - 3n + 4n - 12$

Combine like terms. $= n^2 + n - 12$

To factor a polynomial, first find the greatest common factor (GCF) of the terms. Then use the distributive property to factor out the GCF.

Example 2

Factor $6x^3 - 12x^2 + 18x$.

List the factors of each term. The GCF is $6x$. $6x^3 = 6 \cdot x \cdot x \cdot x$; $-12x^2 = 6 \cdot (-2) \cdot x \cdot x$; $18x = 6 \cdot 3 \cdot x$

Use the distributive property to factor out $6x$. $6x^3 - 12x^2 + 18x = 6x(x^2) + 6x(-2x) + 6x(3)$

$= 6x(x^2 - 2x + 3)$

When a polynomial is the product of two binomials, you can work backward to find the factors.

$$x^2 + bx + c = (x + \blacksquare)(x + \blacksquare)$$

The *sum* of these numbers must equal *b*.

The *product* of these numbers must equal *c*.

Example 3

Factor $x^2 - 13x + 36$.

Choose numbers that are factors of 36. Look for a pair with the sum -13.

The numbers -4 and -9 have a product of 36 and a sum of -13. The factors are $(x - 4)$ and $(x - 9)$. So, $x^2 - 13x + 36 = (x - 4)(x - 9)$.

Factors	Sum
$-6 \cdot (-6)$	-12
$-4 \cdot (-9)$	-13

Need help on factoring and operations with polynomials? Click here.

Exercises

Perform the indicated operations.

1. $(x^2 + 3x - 1) + (7x - 4)$
2. $(5y^2 + 7y) - (3y^2 + 9y - 8)$
3. $4x^2(3x^2 - 5x + 9)$
4. $-5d(13d^2 + 7d + 8)$
5. $(x - 5)(x + 3)$
6. $(n - 7)(n - 2)$

Factor each polynomial.

7. $a^2 - 8a + 12$
8. $n^2 - 2n - 8$
9. $x^2 + 5x + 4$
10. $3m^2 - 9$
11. $y^2 + 5y - 24$
12. $s^3 + 6s^2 + 11s$
13. $2x^3 + 4x^2 - 8x$
14. $y^2 - 10y + 25$

Solving Equations

Plan

How can you isolate the variable?
To isolate the variable, you have to remove the +4 from the left side of the equation.

Problem 1 **Solving a One-Step Equation**

What is the solution of $x + 4 = -12$?

Write the original equation.	$x + 4 = -12$
Use the Subtraction Property of Equality.	$x + 4 - 4 = -12 - 4$
Simplify.	$x = -16$

Subtraction is the inverse operation of addition, so subtract 4 from each side.

Check Substitute -16 for x in the original equation. $-16 + 4 \stackrel{?}{=} -12$

The solution checks. $-12 = -12$ ✔

Plan

How do you solve an equation with the variable on both sides?
Choose a side for the variable and remove it from the other side.

Problem 2 **Solving a Multi-Step Equation**

What is the solution of $-27 + 6y = 3(y - 3)$?

Write the original equation.	$-27 + 6y = 3(y - 3)$
Use the Distributive Property.	$-27 + 6y = 3y - 9$
Add 27 to each side.	$6y = 3y + 18$
Subtract $3y$ from each side.	$3y = 18$
Divide each side by 3.	$y = 6$

Need help on solving equations? Click here.

Exercises

Solve each equation. See Problem 1.

10. $h - 12 = 6$ **11.** $-\frac{x}{3} = 27$ **12.** $4t = 48$

Solve each equation. Check your answer. See Problem 2.

13. $7w + 2 = 3w + 94$

To start, subtract 2 from each side. $7w + 2 - 2 = 3w + 94 - 2$

14. $15 - g = 23 - 2g$ **15.** $5y + 1.8 = 4y - 3.2$

16. $6a - 5 = 4a + 2$ **17.** $4y - 8 - 2y + 5 = 0$

18. $6(n - 4) = 3n$ **19.** $5(2 - g) = 0$

Completing the Square

take note

Key Concept Completing the Square

You can form a perfect square trinomial from $x^2 + bx$ by adding $\left(\frac{b}{2}\right)^2$.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Think

Why do you want a perfect square trinomial?
You can factor a perfect square trinomial as the square of a binomial.

Problem 4 Completing the Square

What value completes the square for $x^2 - 10x$? Justify your answer.

Identify $b = -10$.	$x^2 - 10x$
Find $\left(\frac{b}{2}\right)^2$.	$\left(\frac{b}{2}\right)^2 = \left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$
Add the value of $\left(\frac{b}{2}\right)^2$ to complete the square.	$x^2 - 10x + 25$
Rewrite as the square of a binomial.	$x^2 - 10x + 25 = (x - 5)^2$

take note

Key Concept Solving an Equation by Completing the Square

1. Rewrite the equation in the form $x^2 + bx = c$. To do this, get all terms with the variable on one side of the equation and the constant on the other side. Divide all the terms of the equation by the coefficient of x^2 (if it is not 1).
2. Complete the square by adding $\left(\frac{b}{2}\right)^2$ to each side of the equation.
3. Factor the trinomial.
4. Find square roots.
5. Solve for x.

Problem 5 Solving by Completing the Square

What is the solution of $3x^2 - 12x + 6 = 0$?

Write the original equation.	$3x^2 - 12x + 6 = 0$
Rewrite the equation in the form $x^2 + bx = c$.	$3x^2 - 12x = -6$
Divide each side by 3 so the coefficient of x^2 will be 1.	$\frac{3x^2}{3} - \frac{12x}{3} = \frac{-6}{3}$
Simplify. Identify $b = -4$.	$x^2 - 4x = -2$
Find $\left(\frac{b}{2}\right)^2 = 4$.	$\left(\frac{b}{2}\right)^2 = \left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$
Add 4 to each side.	$x^2 - 4x + 4 = -2 + 4$
Factor the trinomial.	$(x - 2)^2 = 2$
Find square roots.	$x - 2 = \pm\sqrt{2}$
Solve for x.	$x = 2 \pm \sqrt{2}$

Think

How can you check your answer?
Check your results on your calculator. Replace x in the original equation with $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

Completing the Square (cont'd)

Need help on completing the square? Click here.

Exercises

Complete the square.

See Problem 4.

9. $x^2 + 18x + \blacksquare$ **10.** $x^2 - x + \blacksquare$ **11.** $x^2 - 24x + \blacksquare$

12. $x^2 + 20x + \blacksquare$ **13.** $m^2 - 3m + \blacksquare$ **14.** $x^2 + 4x + \blacksquare$

Solve each quadratic equation by completing the square.

See Problem 5.

15. $x^2 + 6x - 3 = 0$

To start, rewrite the equation. Get all terms with x on one side.

$$x^2 + 6x = 3$$

Find $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = (3)^2 = 9$$

16. $x^2 - 12x + 7 = 0$ **17.** $x^2 + 4x + 2 = 0$ **18.** $x^2 - 2x = 5$

19. $x^2 + 12 = 10x$ **20.** $x^2 - 3x = x - 1$ **21.** $x^2 + 2 = 6x + 4$

22. $2x^2 + 2x - 5 = x^2$ **23.** $4x^2 + 10x - 3 = 0$ **24.** $9x^2 - 12x - 2 = 0$

Simplifying Radicals

A radical expression is in simplest form when all of the following are true.

- The radicand has no perfect square factors other than 1.
- The radicand does not contain a fraction.
- A denominator does not contain a radical expression.

radical symbol
$\sqrt{a}$ ← radicand

Example 1

Simplify the expressions $\sqrt{2} \cdot \sqrt{8}$ and $\sqrt{294} \div \sqrt{3}$.

$\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8}$ Write both numbers under one radical.

$= \sqrt{16}$ Simplify the expression under the radical.

$= 4$ Factor out perfect squares and simplify.

$$\sqrt{294} \div \sqrt{3} = \sqrt{\frac{294}{3}}$$
$$= \sqrt{98}$$
$$= \sqrt{49 \cdot 2}$$
$$= 7\sqrt{2}$$

Example 2

Write $\sqrt{\frac{4}{3}}$ in simplest form.

$\sqrt{\frac{4}{3}} = \frac{\sqrt{4}}{\sqrt{3}}$ Rewrite the single radical as a quotient.

$= \frac{2}{\sqrt{3}}$ Simplify the numerator.

$= \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ Multiply by $\frac{\sqrt{3}}{\sqrt{3}}$ (a form of 1) to remove the radical from the denominator. This is called *rationalizing the denominator*.

$= \frac{2\sqrt{3}}{3}$

Some help on this here.

Simplify each radical expression.

1. $\frac{\sqrt{27}}{\sqrt{81}}$

2. $\sqrt{\frac{25}{4}}$

3. $\sqrt{\frac{50}{9}}$

4. $\frac{\sqrt{72}}{\sqrt{50}}$

5. $\sqrt{25} \cdot \sqrt{4}$

6. $\sqrt{27} . \sqrt{3}$

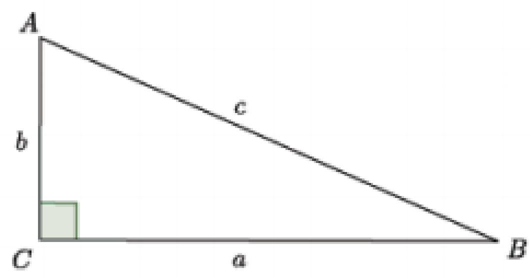
7. $\sqrt{\frac{44x^4}{11}}$

8. $\frac{\sqrt{3c^2}}{\sqrt{27}}$

9. $\sqrt{45} \cdot \sqrt{18}$

Pythagorean's Theorem and Converse

Given a right triangle ABC with C being the vertex of the right angle, then the sides $\overline{AC}$ and $\overline{BC}$ are called the *legs* of $\triangle ABC$, and $\overline{AB}$ is called the *hypotenuse* of $\triangle ABC$.



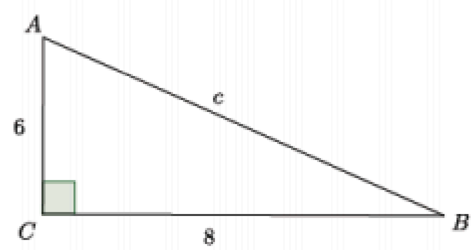
Take note of the fact that side a is opposite the angle A, side b is opposite the angle B, and side c is opposite the angle C.

The Pythagorean theorem states that for any right triangle, $a^2 + b^2 = c^2$.

Example 1

Now that we know what the Pythagorean theorem is, let's practice using it to find the length of a hypotenuse of a right triangle.

Determine the length of the hypotenuse of the right triangle.



The Pythagorean theorem states that for right triangles $a^2 + b^2 = c^2$, where a and b are the legs, and c is the hypotenuse. Then,

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ 100 &= c^2. \end{aligned}$$

Since we know that $100 = 10^2$, we can say that the hypotenuse c is 10.

The converse of the Pythagorean theorem states that if a triangle with side lengths a, b, and c satisfies $a^2 + b^2 = c^2$, then the triangle is a right triangle.

For more help on Pythagorean and Converse, click here.

Exercises

Determine whether the given lengths can be side lengths of a right triangle.

1. 15, 36, 39
2. 3, 7, 10
3. 8, 15, 17
4. $\sqrt{3}, \sqrt{4}, \sqrt{5}$
5. 6, 7, 8
6. 12, 16, 20

For the values given, *a* and *b* are legs of a right triangle. Find the length of the hypotenuse. If necessary, round to the nearest tenth.

7. $a = 6$, $b = 8$
8. $a = 5$, $b = 9$
9. $a = 4$, $b = 10$
10. $a = 9$, $b = 1$
11. $a = 7$, $b = 3.5$
12. $a = 1.4$, $b = 2.3$

Right Triangle Trigonometry

For any right triangle, there are six trig ratios: Sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot).

Here are the formulas for these six trig ratios:

$$\sin = \frac{opp}{hyp} \qquad \csc = \frac{1}{\sin} = \frac{hyp}{opp}$$

$$\cos = \frac{adj}{hyp} \qquad \sec = \frac{1}{\cos} = \frac{hyp}{adj}$$

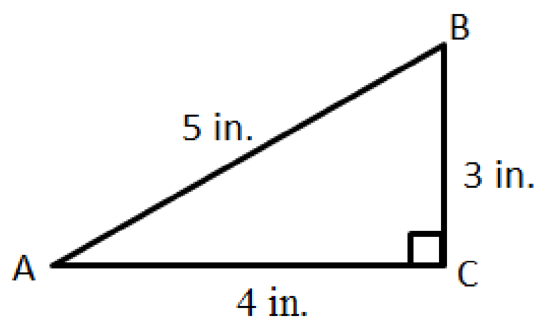
$$\tan = \frac{opp}{adj} \qquad \cot = \frac{1}{\tan} = \frac{adj}{opp}$$

Let's start by finding all 6 ratios for angle A.

$$\sin \mathrm{A} = \frac{opp}{hyp} = \frac{3}{5} \qquad \csc \mathrm{A} = \frac{1}{\sin} = \frac{5}{3}$$

$$\cos \mathrm{A} = \frac{adj}{hyp} = \frac{4}{5} \qquad \sec \mathrm{A} = \frac{1}{\cos} = \frac{5}{4}$$

$$\tan \mathrm{A} = \frac{opp}{adj} = \frac{3}{4} \qquad \cot \mathrm{A} = \frac{1}{\tan} = \frac{4}{3}$$

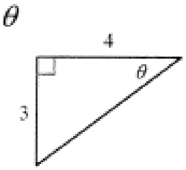


Help on trig ratios can be found here.

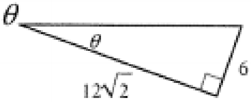
Exercises

Find the value of each indicated trigonometric ratio.

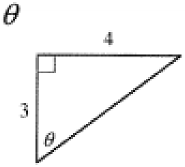
1) $\cot \theta$



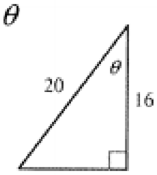
2) $\csc \theta$



3) $\sin \theta$



4) $\csc \theta$



Right Triangle Trigonometry Exercises (cont'd)

Write the ratio of the indicated trig functions:

5) Find $\sec \theta$ if $\tan \theta = \frac{\sqrt{5}}{2}$

6) Find $\csc \theta$ if $\tan \theta = \frac{7\sqrt{15}}{60}$

7) Find $\sec \theta$ if $\cot \theta = 2$

8) Find $\cos \theta$ if $\cot \theta = \frac{8}{15}$

9) Find $\cos \theta$ if $\sec \theta = \frac{5\sqrt{2}}{7}$

10) Find $\cos \theta$ if $\csc \theta = \sqrt{5}$